Chapter 2
Lorentz contraction from the classical wave equation
–
from my book:
Understanding Relativistic Quantum Field Theory

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Contents

Chapter 2

Lorentz contraction from the classical wave equation

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2.1 Moving solutions of the classical wave equation

We turn our attention again to the classical wave equation. This time to look at solutions which are moving with a constant speed v, say for instance in the x-direction. An arbitrary function which *shifts* along with a speed v in the x direction does satisfy equations which relates the derivatives in t and x by the speed v in the following way:

$$\frac{\partial \Phi}{\partial t} = -v\frac{\partial \Phi}{\partial x} \qquad \frac{\partial^2 \Phi}{\partial t^2} = v^2\frac{\partial^2 \Phi}{\partial x^2} \tag{2.1}$$

These equations are valid for any arbitrary potential function Φ. We can combine the latter in the the the classical wave equation for three spatial dimensions.

$$\frac{\partial^2 \Phi}{\partial t^2} - c^2\frac{\partial^2 \Phi}{\partial x^2} - c^2\frac{\partial^2 \Phi}{\partial y^2} - c^2\frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{2.2}$$

and use it to replace the 2nd order time derivative with the 2nd order x-derivative in order to eliminate the time dependency. We get:

$$\left(1 - \frac{v^2}{c^2}\right)c^2\frac{\partial^2 \Phi}{\partial x^2} + c^2\frac{\partial^2 \Phi}{\partial y^2} + c^2\frac{\partial^2 \Phi}{\partial z^2} = 0 \tag{2.3}$$

This shows that the solutions are Lorentz contracted in the direction of v by a factor γ, The first order derivatives are higher by a factor γ and the second order by a factor γ^2. Velocities higher then c are not possible. This proof can't hardly be any simpler, however we want to study this in some more detail by using the 3d-propagator.

Figure 2.1 shows how the field Φ propagates away from the charge spherically while decreasing in amplitude $1/r$. Thicker circles depict a higher amplitude. The field behind the charge was emitted more recently, the "circles" have a higher amplitude but are further separated. The field in front of the charge was emitted longer ago, the circles have a lower amplitude but they are compressed closer together.

Since the Lorentz contracted field is mirror-symmetric in the x-axis, we conclude that the effect of the higher/lower amplitude apparently must be

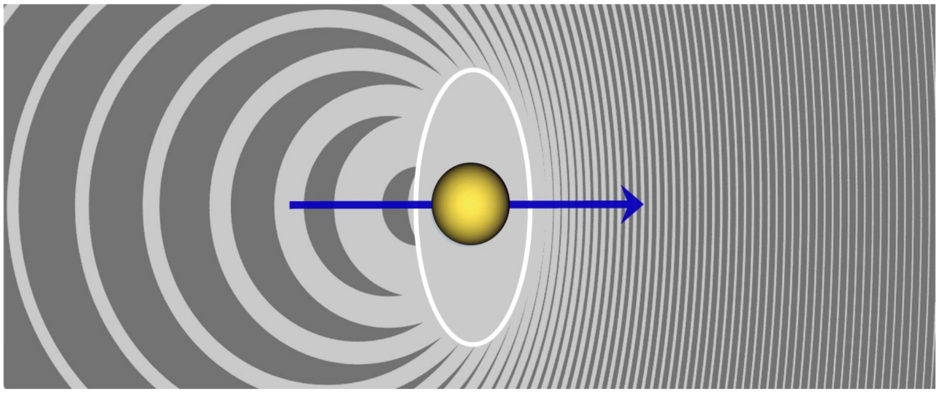

Figure 2.1: Electrostatic potential of a charge moving at 0.8 c

compensated by the effect of separation/compression. With this in mind we can study the Liénard Wiechert potentials.

2.2 The Liénard Wiechert potentials

Alfred Liénard (in 1898) and Emil Wiechert (in 1900) determined the potentials of an arbitrarily moving and accelerating point charge. They are based on the assumption that the potentials spread from the source with the speed of light with an attenuation of $1/r$ as we found when we derived the propagator of the wave-function for three spatial dimensions. The equations for the electric potential Φ and the magnetic vector potential A are close to the classical ones:

$$\Phi(\vec{x},t) = \frac{q}{4\pi\epsilon_0 r_{ret}} \left[\frac{1}{(1-\beta\cos\phi)} \right] \tag{2.4}$$

$$\vec{A}(\vec{x},t) = \frac{q}{4\pi\epsilon_0 r_{ret}} \left[\frac{\vec{\beta}}{(1-\beta\cos\phi)} \right] \tag{2.5}$$

$$\text{Where:} \begin{cases} \vec{\beta} & = & \text{speed vector of the charge: } \vec{v}/c \\ r_{ret} & = & \text{distance from the retarded charge.} \\ 1/(1-\beta\cos\phi) & = & \text{compression or 'shockwave' factor} \end{cases}$$

Note that the equations use the retarded charge location: The location where the charge was at the moment when the potentials were emitted from the point charge. From the formula's we see that all the information that is needed is:

(1) the charge, (2) its location, and (3) its speed.

We have this information if we know the position of the point charge at two different moments in time, at t and at t+dt. The value of Φ at a single point in space-time (t,r) therefor doesn't contain any information about the acceleration of the charge, it only depends on position and speed. This is unlike the equations for the E and B fields which do in fact depend on the acceleration, because they are based on the derivatives of the potential fields.

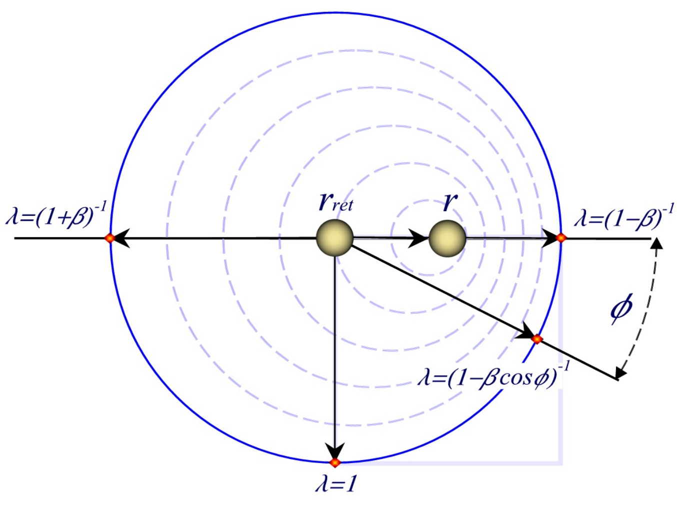

Figure 2.2: The shockwave factor under various angles

The only thing which needs some explanation is the compression or 'shockwave' factor. We concluded that this effect should occur in the previous section on Lorentz contraction. It can become infinite in front of the moving charge (psi = 0) when the speed goes to c. This effect is equal to the shockwave building up in front of a plane which approaches the speed of

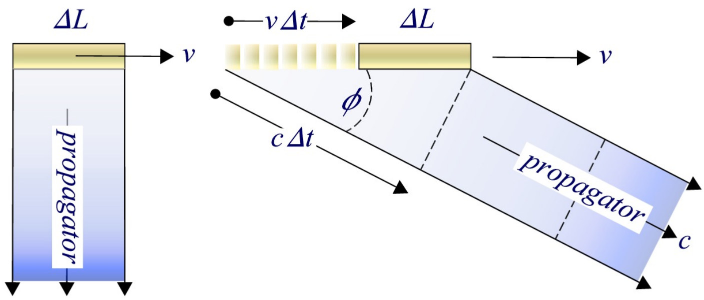

Figure 2.3: The shockwave factor λ is $1/(1-\beta\cos\phi)$

sound. The smallest it can get is 1/2. This is behind the charge when it is moving at c.

From figure 2.3 we see that the propagation under a non-transversal angle received by an observer is emitted at different moments. The propagation from the tail travels $c\Delta t$ longer as the propagation from the front. This effectively stretches the length of the charge ΔL to $\Delta L + v\Delta t$. The effective length of the charge relates to the extra time as:

$$(v\Delta t + \Delta L)\cos\phi \;=\; c\Delta t \qquad \Rightarrow \qquad \Delta t = \frac{\Delta L\cos\phi}{c - v\cos\phi} \tag{2.6}$$

Having obtained an expression for Δt we can calculate the effective length. The relative increase of the length is equal to the relative increase of the potentials fields due to the shock factor. For the increased length $\Delta L'$ we get:

$$\Delta L' = \Delta L + v\Delta t = \Delta L + \frac{v\Delta L\cos\phi}{c - v\cos\phi} = \frac{\Delta L}{1 - \frac{v}{c}\cos\phi} = \frac{\Delta L}{1 - \vec{\beta}\cdot\hat{r}_{ret}} \tag{2.7}$$

$$\text{Shockwave factor:} \;=\; \frac{1}{1-\beta\cos\phi} \;=\; \frac{1}{1-\vec{\beta}\cdot\hat{r}_{ret}} \tag{2.8}$$

2.3 The Lorentz contracted EM potentials

To obtain the Lorentz contracted fields from the Liénard Wiechert potentials we need to rewrite them with the respect to the current location of the charge instead of the retarded using the fact that the velocity is now constant. If we locate the current position at the origin then we can derive the distance from any point (x, y, z) to the retarded location and the angle ϕ belonging to that location.

$$r_{ret} \;=\; \sqrt{(x+vt)^2+y^2+z^2}, \qquad \cos\phi \;=\; \frac{(x+vt)}{\sqrt{(x+vt)^2+y^2+z^2}} \tag{2.9}$$

Checking this from positions on the three principle axis we find for the Lorentz contracted potential.

$$\begin{array}{lclcl} (x,0,0) & \Rightarrow & r_{ret} = x/(1-\beta), & \cos\phi & = & \pm 1 \\ (0,y,0) & \Rightarrow & r_{ret} = \gamma\; y, & \cos\phi & = & \beta \\ (0,0,z) & \Rightarrow & r_{ret} = \gamma\; z, & \cos\phi & = & \beta \end{array} \tag{2.10}$$

giving the potential fields on the principle axis:

$$\Phi(x,0,0) = \frac{q}{4\pi\epsilon_0 x}, \qquad \vec{A}(x,0,0) = \frac{\vec{\beta}\; q/c}{4\pi\epsilon_0 x} \tag{2.11}$$

$$\Phi(0,y,0) = \frac{\gamma\; q}{4\pi\epsilon_0 y}, \qquad \vec{A}(0,y,0) = \frac{\gamma\vec{\beta}\; q/c}{4\pi\epsilon_0 y} \tag{2.12}$$

$$\Phi(0,0,z) = \frac{\gamma\; q}{4\pi\epsilon_0 z}, \qquad \vec{A}(0,0,z) = \frac{\gamma\vec{\beta}\; q/c}{4\pi\epsilon_0 z} \tag{2.13}$$

The general expressions for the Lorentz contracted potentials for a charge moving on the x-axis:

$$\Phi(x,y,z) \;=\; \frac{\gamma\; q}{4\pi\epsilon_0\sqrt{(\gamma x)^2+y^2+z^2}} \tag{2.14}$$

$$\vec{A}(x,y,z) \;=\; \frac{\gamma\;\vec{\beta}\; q/c}{4\pi\epsilon_0\sqrt{(\gamma x)^2+y^2+z^2}} \tag{2.15}$$

Which is indeed what we would expect from the previous section. The potential fields are Lorentz contracted by a factor γ while the amplitude of the potentials are also multiplied by a factor γ. The result is that the potentials in front and behind the moving charge are equal to those of a charge at rest, while the potentials at 90 degrees angles with the velocity are higher by a factor γ.

2.4 The Lorentz transform of the EM potentials

We can now derive the general Lorentz transform of the EM potential field. We can then check the transform with what we derived in the previous section: Φ and $\vec{A}$ of a moving charge. We did so with the help of the Liènard Wiechert potentials. We did derive the LW potentials from the propagator of the field which in turn we derived from the classical wave equation. We can combine Φ and $\vec{A}$ into a single four vector $A^\mu = \{\Phi/c\,,\vec{A}\}$ (in the SI-unit system). Often we will omit c by defining it to be 1.

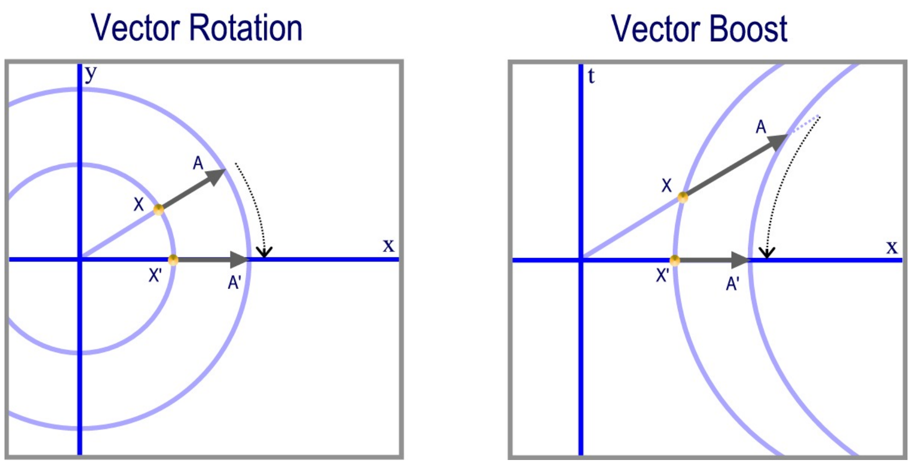

Figure 2.4: Rotation and boost of a vector field

The Lorentz transform of the electromagnetic potential A^μ is the transform of a (four-) vector field. The simplest transform of a field is that of a (Lorentz) scalar field. A scalar has the same value in every reference frame, all we have to do is a Lorentz coordinate transformation and the new gives us the scalar value at the new positions.

In case of a vector field we also need to transform the vector itself as shown in figure 2.4. In this case we can simply treat the four vector A^μ as the four vector x^μ. The standard Lorentz coordinate transform (with $c = 1$) is given by.

forward transform | backward transform

$$\begin{aligned} t' &= \gamma\left(t - \vec{\beta}\cdot\vec{x}\right) & t &= \gamma\left(t' + \vec{\beta}\cdot\vec{x}'\right) \\ \vec{x}'_{\parallel} &= \gamma\left(\vec{x}_{\parallel} - \vec{\beta}\, t\right) & \vec{x}_{\parallel} &= \gamma\left(\vec{x}'_{\parallel} + \vec{\beta}\, t'\right) \\ \vec{x}'_{\perp} &= \vec{x}_{\perp} & \vec{x}_{\perp} &= \vec{x}'_{\perp} \end{aligned} \tag{2.16}$$

Where we have split $\vec{x}$ into $\vec{x}_{\parallel}+\vec{x}_{\perp}$, the components parallel and orthogonal to the boost β. We now simply replace x^μ with A^μ to obtain the Lorentz transform of the potential field.

Lorentz transformation of the electromagnetic four-vector A^μ

forward transform | backward transform

$$\begin{aligned} \Phi' &= \gamma\left(\Phi - \vec{\beta}\cdot\vec{A}\right) & \Phi &= \gamma\left(\Phi' + \vec{\beta}\cdot\vec{A}'\right) \\ \vec{A}'_{\parallel} &= \gamma\left(\vec{A}_{\parallel} - \vec{\beta}\, \Phi\right) & \vec{A}_{\parallel} &= \gamma\left(\vec{A}'_{\parallel} + \vec{\beta}\, \Phi'\right) \\ \vec{A}'_{\perp} &= \vec{A}_{\perp} & \vec{A}_{\perp} &= \vec{A}'_{\perp} \end{aligned} \tag{2.17}$$

Where we have used c=1 (for full SI replace Φ by Φ/c). The parallel and orthogonal components can be given as vector expressions involving the unit-vector $\hat{\beta}$ in the direction of the boost.

$$\begin{aligned} \vec{A}_{\parallel} &= (\,\hat{\beta}\cdot\vec{A}\,)\,\hat{\beta} && \text{parallel component with regard to } \vec{\beta} \\ \vec{A}_{\perp} &= (\,\hat{\beta}\times\vec{A}\,)\times\hat{\beta} && \text{orthogonal component with regard to } \vec{\beta} \end{aligned} \tag{2.18}$$

In the simple case of a transformation from the rest-frame of the source charge of the field to a boosted frame we can write.

$$\left(\Phi/c,\ \vec{A} \right) \quad \text{transforms like} \quad \left(\gamma,\ \vec{\beta}\gamma \right) \tag{2.19}$$

We see this back in the general expressions for the Lorentz contracted potentials for a charge moving on the x-axis which we derived in section 2.3.

$$\Phi(x,y,z) \ = \ \frac{\gamma\ q}{4\pi\epsilon_0\sqrt{(\gamma x)^2+y^2+z^2}} \tag{2.20}$$

$$\vec{A}(x,y,z) \ = \ \frac{\gamma\ \vec{\beta}\ q/c}{4\pi\epsilon_0\sqrt{(\gamma x)^2+y^2+z^2}} \tag{2.21}$$

The changes $x \rightarrow \gamma x$ are the result of the Lorentz contraction (the co-ordinate transformation) while the factors γ and $\vec{\beta}\gamma$ are the result of the transformation of the transformation of the four-vector A^μ.

2.5 The Lorentz transform of charge and current

Since the electromagnetic potential A^μ has as its source the charge/current density $j^\mu = \{\ \rho\ ,\vec{j}\ \}$ we may expect that A^μ and j^μ transform in the same way.

Lorentz transformation of the charge-current density j^μ

forward transform — backward transform

$$\begin{array}{lll} \rho' & = & \gamma\left(\rho\ - \vec{\beta}\cdot\vec{j}\ \right) \\ \vec{j}'_{\parallel} & = & \gamma\left(\vec{j}_{\parallel}\ - \vec{\beta}\ \rho\ \right) \\ \vec{j}'_{\perp} & = & \vec{j}_{\perp} \end{array} \qquad \begin{array}{lll} \rho & = & \gamma\left(\rho'\ + \vec{\beta}\cdot\vec{j}'\ \right) \\ \vec{j}_{\parallel} & = & \gamma\left(\vec{j}'_{\parallel}\ + \vec{\beta}\ \rho'\ \right) \\ \vec{j}_{\perp} & = & \vec{j}'_{\perp} \end{array} \tag{2.22}$$

Where the parallel and orthogonal components of $\vec{j}$ relative to the boost are given by.

$$\begin{array}{lcll} \vec{j}_{\parallel} & = & (\,\hat{\beta}\,\cdot\,\vec{j}\,)\,\hat{\beta} & \text{parallel component with regard to } \vec{\beta} \\ \vec{j}_{\perp} & = & (\,\hat{\beta}\times\vec{j}\,)\times\hat{\beta} & \text{orthogonal component with regard to } \vec{\beta} \end{array} \tag{2.23}$$

A charge density at rest transforms into a charge/current density as.

$$\Big(\,\rho\ ,\ \vec{j}\,\Big) \ \text{ transforms like } \ \Big(\,\gamma\ ,\ \vec{\beta}\gamma\,\Big) \tag{2.24}$$

The total charge Q and total current $\vec{\mathbf{J}}$

For sofar we have discussed the transformation of the charge and current *densities*. The total charge and total current are obtained by integrating over space. The space over which the charge/current is spread reduces by a factor gamma, with the result that the total charge and current transform less by a factor γ, thus.

$$\Big(\,Q\ ,\ \vec{J}\,\Big) \ \text{ transforms like } \ \Big(\,1\ ,\ \vec{\beta}\,\Big) \tag{2.25}$$

This is a fundamental result. It means that the charge is *reference frame independent*. The current is always proportional to the speed. This in contrast with the energy/momentum of a particle which transforms as we know like.

$$\left(\,E\ ,\ \vec{p}\,\right) \ \text{ transforms like } \ \Big(\,\gamma\ ,\ \vec{\beta}\gamma\,\Big) \tag{2.26}$$

Again we encounter something truly fundamental here. The energy/ momentum of a particle determines its resistance to the change of motion due to a force exerted on the particle. The electromagnetic force is determined by the value of the charge Q^2, which in contrast to the "relativistic mass" γmc^2 does not transform.

The difference of the factor γ in the way which E and Q^2 transform now leads to the effect of *time dilatation* whereby all processes proceed slower by a factor γ. We will discuss this subject in more detail in chapter **??**: "Time dilation from the classical wave equation".

2.6 The Lorentz contracted E and B fields

We want to derive the E and B fields of a moving electric charge. This is straightforwardly done by using Maxwell's laws to obtain the field from the potentials Φ and $\vec{A}$. The general expressions for the Lorentz contracted potentials for a charge moving on the x-axis are:

$$\Phi(x,y,z) \;=\; \frac{\gamma\; q}{4\pi\epsilon_0\sqrt{(\gamma x)^2+y^2+z^2}} \tag{2.27}$$

$$\vec{A}(x,y,z) \;=\; \frac{\gamma\; \vec{\beta}\; q/c}{4\pi\epsilon_0\sqrt{(\gamma x)^2+y^2+z^2}} \tag{2.28}$$

Where $\vec{\beta} = \{\ \beta_x\ ,\ 0\ ,\ 0\ \}$ is along the x-axis. The E and B fields are derived from the potentials by.

$$\mathsf{E} \;=\; -\text{grad}\ \Phi - \frac{\partial \vec{A}}{\partial t} \tag{2.29}$$

$$\mathsf{B} \;=\; \text{curl}\ \vec{A} \tag{2.30}$$

When written out in full these give us.

$$\mathsf{E} \;=\; \left\{ -\frac{\partial\Phi}{\partial x} - \frac{\partial A_x}{\partial t}\ ,\ -\frac{\partial\Phi}{\partial y} - \frac{\partial A_y}{\partial t}\ ,\ -\frac{\partial\Phi}{\partial z} - \frac{\partial A_z}{\partial t} \right\} \tag{2.31}$$

$$\mathsf{B} \;=\; \left\{ \ \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\ ,\ \ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\ ,\ \ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right\} \tag{2.32}$$

The magnetic vector potential components A_y and A_z are zero while A_x has a simple relation with the potential Φ:

$$A_x \;=\; \beta_x \Phi \tag{2.33}$$

We can change a derivative in t to derivative in x by simply multiplying it with $-\beta_x$ since our solution shifts in the x-direction with a speed v_x, so we have:

$$\frac{\partial A_x}{\partial t} \;=\; -\beta_x \frac{\partial A_x}{\partial x} \;=\; -\beta_x^2 \frac{\partial \Phi}{\partial x}, \qquad \text{thus:} \tag{2.34}$$

$$E_x \;=\; -\frac{\partial \Phi}{\partial x} \;-\; \frac{\partial A_x}{\partial t} \;=\; -(1-\beta_x^2)\frac{\partial \Phi}{\partial x} \tag{2.35}$$

This gives us for the E_x component of the electric field.

$$E_x \;=\; \frac{(1-\beta_x^2)\ \gamma^3 x\ q}{4\pi\epsilon_o(\gamma^2x^2+y^2+z^2)^{3/2}} \tag{2.36}$$

$$E_x \;=\; \frac{\gamma\ x\ q}{4\pi\epsilon_o(\gamma^2x^2+y^2+z^2)^{3/2}} \tag{2.37}$$

The E_y and E_z components are simpler since $A_y = A_z = 0$. These components of the electric field become.

$$E_y = \frac{\gamma\ y\ q}{4\pi\epsilon_o(\gamma^2x^2+y^2+z^2)^{3/2}}, \quad E_z = \frac{\gamma\ z\ q}{4\pi\epsilon_o(\gamma^2x^2+y^2+z^2)^{3/2}} \tag{2.38}$$

We see that the factor $(1-\beta_x^2)$ was canceled by the extra factor γ^2 due to the differentiation along the (Lorentz contracted) x-axis. All nominators have a similar form now so we can simply write this in vector form.

Electric field E of a moving charge

$$\mathsf{E} \;=\; \frac{\gamma\ \vec{r}\ q}{4\pi\epsilon_o(\gamma^2x^2+y^2+z^2)^{3/2}} \tag{2.39}$$

When we derive the magnetic field B we see that the the E field and the B field relate to each other in a simple way.

$$\mathsf{E} \;=\; -\vec{v}\times\mathsf{B}, \qquad \mathsf{B} \;=\; \frac{\vec{v}}{c^2}\times\mathsf{E} \tag{2.40}$$

So here we can obtain the magnetic field B simple from the electric field.

Magnetic field B of a moving charge

$$\mathsf{B} \;=\; \frac{\gamma\ \left(\vec{\beta}\times\vec{r}\right)\ q/c}{4\pi\epsilon_o(\gamma^2x^2+y^2+z^2)^{3/2}} \tag{2.41}$$

Relation 2.40 holds for an arbitrary moving monopole point source. In the limit cases for $\beta = 0$ we retrieve the standard static fields.

$$\lim_{\beta\to 0} \mathsf{E} \;=\; \frac{\vec{r}\, q}{4\pi\epsilon_o r^3}, \qquad\qquad \lim_{\beta\to 0} \mathsf{B} \;=\; \frac{\vec{\beta}\times\vec{r}\; q/c}{4\pi\epsilon_o r^3} \tag{2.42}$$

We can represent these expressions in spherical coordinates with the help of the replacement $\sin^2\theta \;= (y^2+z^2)/(x^2+y^2+z^2)$

$$\mathsf{E} \;=\; \frac{\vec{r}\, q}{4\pi\epsilon_o\; r^3\gamma^2(1-\beta^2\sin^2\theta)^{3/2}} \tag{2.43}$$

$$\mathsf{B} \;=\; \frac{\left(\vec{\beta}\times\vec{r}\right)\; q/c}{4\pi\epsilon_o\; r^3\gamma^2(1-\beta^2\sin^2\theta)^{3/2}} \tag{2.44}$$

2.7 Lorentz transform of the E and B fields

In the previous section we made use of a relation between the electric and magnetic fields of a moving (monopole) charge.

$$\mathsf{E} \;=\; -\vec{v}\times\mathsf{B}, \qquad \mathsf{B} \;=\; \frac{\vec{v}}{c^2}\times\mathsf{E} \tag{2.45}$$

We can prove this easily for a constant speed v with the help of the generic expressions.

$$\mathsf{E} \;=\; -\mathrm{grad}\;\Phi - \frac{\partial\vec{A}}{\partial t}, \qquad\qquad \mathsf{B} \;=\; \mathrm{curl}\;\vec{A} \tag{2.46}$$

Substitution gives us for the magnetic field.

$$\mathsf{B} \;=\; \mathrm{curl}\;\vec{A} \;=\; -\,\frac{\vec{v}}{c^2}\times\mathrm{grad}\;\Phi \;-\; \frac{\vec{v}}{c^2}\times\frac{\partial\vec{A}}{\partial t} \tag{2.47}$$

The last term with the time derivative is zero if the speed v is constant due to the relation $\vec{A} = \Phi\vec{v}/c^2$. The vector potential $\vec{A}$ points always in the direction of $\vec{v}$, so a change of $\vec{A}$ is also in the direction of $\vec{v}$ and the cross product is always zero.

$$\frac{\vec{v}}{c^2} \times \frac{\partial \vec{A}}{\partial t} \;=\; 0 \tag{2.48}$$

Without this term and after replacing $\vec{A}$ with $\Phi\vec{v}/c^2$ the remaining expression becomes.

$$\mathsf{B} \;=\; \frac{1}{c^2}\; \nabla \times (\Phi\vec{v}) \;=\; \frac{1}{c^2}\; \nabla(\Phi) \;\times\; \vec{v} \tag{2.49}$$

Which is true due to the chain rule and $\nabla \times \vec{v} = 0$. in the situation of an arbitrary charge/current density distribution this relation between E and B obviously doesn't hold, however, it does appears in the Lorentz transform of the electromagnetic field.

In the case of the Lorentz transform of the EM-field under a (constant) boost in velocity the relations (2.77) determine how the electric and magnetic field are transformed into each other. For instance in the case of a boost in the x-direction.

$$\begin{array}{rclcrcl} E'_x &=& E_x & \qquad & B'_x &=& B_x \\ E'_y &=& \gamma(E_y - vB_z) & & B'_y &=& \gamma(B_y + \frac{v}{c^2}E_z) \\ E'_z &=& \gamma(E_z + vB_y) & & B'_z &=& \gamma(B_z - \frac{v}{c^2}E_y) \end{array} \tag{2.50}$$

Which we can rewrite for an arbitrary boost using a short hand notation for the various components of the fields parallel and orthogonal with regard to the boost $\vec{\beta}$ and further simplified by setting c to 1.

Lorentz transform of the electromagnetic field

$$\begin{array}{rcl} \mathsf{E}' &=& \mathsf{E}_{\parallel} \;+\; \mathsf{E}_{\perp}\,\gamma \;+\; \mathsf{B}_{\otimes}\,\beta\gamma \\ \mathsf{B}' &=& \mathsf{B}_{\parallel} \;+\; \mathsf{B}_{\perp}\,\gamma \;-\; \mathsf{E}_{\otimes}\,\beta\gamma \end{array} \tag{2.51}$$

$$\begin{array}{rcll} \mathsf{E}_{\parallel} &=& (\,\hat{\beta}\,\cdot\,\mathsf{E}\,)\,\hat{\beta} & \text{parallel component with regard to } \vec{\beta} \\ \mathsf{E}_{\perp} &=& (\,\hat{\beta}\times\mathsf{E}\,)\times\hat{\beta} & \text{orthogonal component with regard to } \vec{\beta} \\ \mathsf{E}_{\otimes} &=& (\,\hat{\beta}\times\mathsf{E}\,) & 90^o \text{ rotated orthogonal component} \end{array} \tag{2.52}$$

We can apply these transformation expressions on a moving point charge and check if we get the same results as in the previous section. First we have to apply a coordinate transform. We are only interested in $t'{=}0$ where the particle is in the coordinate center. The fields of the particle are independent on time in the particles rest-frame. So, al we need to do is replacing x by γx which corresponds to the Lorentz contraction.

$$\mathsf{E} \;=\; \frac{\vec{r}\;q}{4\pi\epsilon_o r^3} \quad\Rightarrow\quad \frac{(\;\gamma x\;\hat{x}\;+\;y\;\hat{y}\;+\;z\;\hat{z}\;)\;q}{4\pi\epsilon_o(\gamma^2x^2+y^2+z^2)^{3/2}} \tag{2.53}$$

After the coordinate transformation we have to do the field transformation. The electromagnetic field transforms, in the way as given above, as an anti-symmetric tensor. which we will discuss in more detail in the chapter on the relativistic formulation of fields.

From the Field transform equations (2.51) we see that the components *orthogonal* to the boost acquire an extra factor γ while the component parallel to the boost doesn't. The total transformation, coordinate plus field transformation thus yields:

$$\mathsf{E} \;=\; \frac{\vec{r}\;q}{4\pi\epsilon_o r^3} \quad\Rightarrow\quad \frac{\gamma\vec{r}\;q}{4\pi\epsilon_o(\gamma^2x^2+y^2+z^2)^{3/2}} \tag{2.54}$$

Which corresponds to expression (2.39) which we derived in the previous section. Now we want to obtain the magnetic field of a moving charge from the general Lorentz transform of the magnetic field. In the rest frame there is no magnetic field but we have to transform the rest-frames E field into a magnetic field.

First step is again the coordinate transform of the E-field as in (2.53) after which comes the field transform. This involves a cross-product $\vec{\beta}\times\mathsf{E}$ which uses only the components orthogonal of E in (2.53) removing the term $\gamma x\;\hat{x}$. The result we get corresponds with the magnetic field of a moving charge which we derived in (2.41).

$$\mathsf{B} \;=\; \frac{\gamma\;\left(\vec{\beta}\times\vec{r}\right)\;q/c}{4\pi\epsilon_o(\gamma^2x^2+y^2+z^2)^{3/2}} \tag{2.55}$$

2.8 The Liénard Wiechert E and B fields

It is important to realize that the potentials Φ and A are propagating on the light cone and the electric and magnetic fields E and B are derivative fields. If we would rewrite the formula's for the E and B fields in the same way using the retarded location and a shockwave factor then we obtain incorrect expressions, which would for instance not show any electromagnetic radiation.

We do however get the EM fields by carefully differentiating the potentials in space and time. Carefully because we use retarded values in our formula's for Φ and A to obtain current values. Before we proceed into this we'll first have a look at the results:

$$\mathsf{E}(\vec{x},t) = \frac{q}{4\pi\epsilon_0 r_{ret}^2}\left[\frac{(1-\beta^2)\ \vec{r}_{ph}}{(1-\beta\cos\phi)^3}\right] + \frac{q}{4\pi\epsilon_0 r_{ret}}\left[\frac{\hat{r}_{ret}\times(\vec{r}_{ph}\times\vec{a})}{c^2(1-\beta\cos\phi)^3}\right] \tag{2.56}$$

$$\mathsf{B}(\vec{x},t) = \frac{\hat{r}_{ret}}{c}\times\mathsf{E} \tag{2.57}$$

$$\text{where: }\begin{cases} \vec{a} & = \text{ accelaration vector of the charge.} \\ \hat{r}_{ret} & = \text{ unit vector from retarded charge towards } (\vec{x},t) \\ \vec{r}_{ph} & = \text{ vector } (\hat{r}_{ret}-\vec{\beta}) \text{ from phantom location to } (\vec{x},t) \end{cases}$$

with $v \ll c$ this simplifies to:

$$\mathsf{E}(\vec{x},t) = \frac{q}{4\pi\epsilon_0 r^2}\ \hat{r} - \frac{q}{4\pi\epsilon_0 r}\ \frac{1}{c^2}\ \hat{r}\times(\hat{r}\times\vec{a}) \tag{2.58}$$

We see that the first term is the standard Coulomb field. The second term is the radiation term which is proportional to the acceleration of the charge. The radiation term decreases with only $1/r$ rather than with $1/r^2$ as the Coulomb field does. This leads to a finite energy flux $(\mathsf{E}\times\mathsf{B})/\mu_o$ away from the charge at any $r\to\infty$.

A constant amount of energy has to be fed to the charge to keep it radiating. This in contrast with the energy associated with the Coulomb field. If we could "create" a charge, (which we can't because charge is a conserved quantity), then the amount of energy we would need to build up the Coulomb field would decrease by $1/t^2$ and reach some maximum in the limit case of t going to infinity.

2.9 The phantom location of a moving charge

An interesting phenomena is the appearance of the 'phantom' location in the general formula. This location is were the charge would be if it would have continued at the same speed after the potentials left the charge at the retarded position. The force from the generalized Coulomb field points to (or from) the phantom location. The phantom location is the real current location for a charge with constant velocity. It therefor might look like the Coulomb field is an instantaneous field since, no matter how far away, the field lines always point to the current location of charge. However, when the charge changes its velocity and its path in the mean time then the phantom location and the real current location no longer coincide.

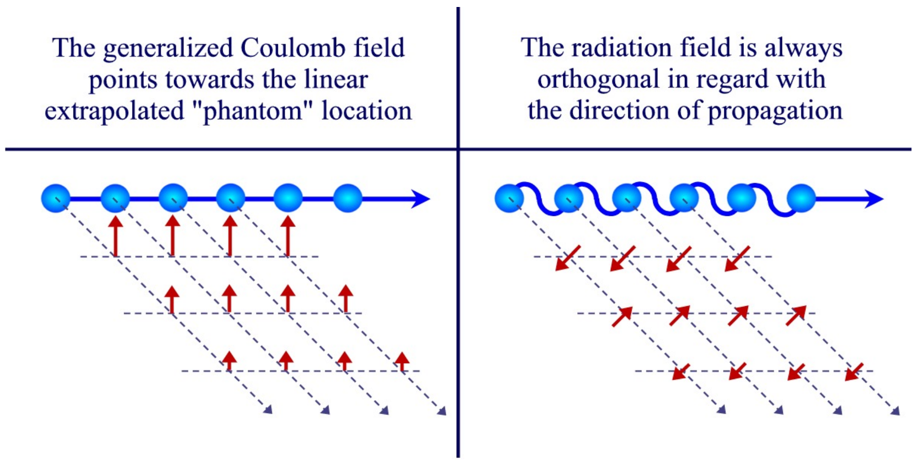

Figure 2.5: The Coulomb field and the radiation field

At the other hand, The E and B components of the *radiation* field are always *orthogonal* to the vector coming from the *retarded* position. The Poynting vector representing the energy flux is always pointing away from the retarded location. By looking at the radiation we can see where the charge was at the moment that it was emitted. We see this effect also if we look at the Sun. The light comes from the retarded position while the gravitation pulls the earth into the direction of the phantom position. (The gravitational field of a moving body behaves similar[1] as the EM field).

[1]The phantom position in GR also involves an extra term making it an even better extrapolation of the current to the future position.

2.10 Derivation of the Liénard Wiechert E field

To derive the electric field E and the magnetic field B from the Liénard Wiechert potentials we have to apply the standard formulas.

$$\mathsf{E} \;=\; -\nabla\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}, \qquad \mathsf{B} \;=\; \nabla\times\vec{A} \tag{2.59}$$

However, we express the Liénard Wiechert potentials, and subsequently the E and B fields, using *retarded* values, like the distance, speed, and acceleration, that the charge *had* at the moment when the potentials were emitted.

The above derivatives using dt and dx should therefor be translated to retarded values like $dt \;\rightarrow\; dt_{ret}$. To avoid too many subscripts we use square bracket that enclose expressions which exist out of purely retarded values. For dt_{ret} we can write:

$$dt_{ret} \;=\; dt \;+\; \Big[\; \vec{\beta}\cdot\hat{r} \;\Big]_{ret}\; dt_{ret} \tag{2.60}$$

Which can be explained as follows: dt corresponds to a delta time at our location where we calculate the E-field, while dt_{ret} is the corresponding delta time at the source. We see that the delta time by which the potentials were emitted *increases* if the charge moves closer towards us.

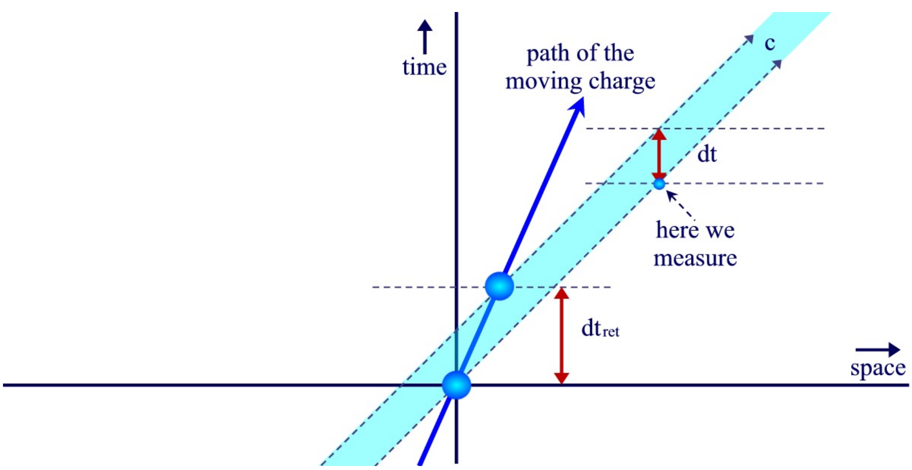

Figure 2.6: Relation between dt and dt_{ret}

It takes a longer time at the beginning of dt_{ret} to propagate from source to us as at the end of dt_{ret} since the charge has moved closer during the time dt_{ret}. We recover the shockwave factor if we write down the ratio between the time deltas:

$$\frac{\partial t_{ret}}{\partial t} = \left[\frac{1}{1-\vec{\beta}\cdot\hat{r}} \right]_{ret} \tag{2.61}$$

We can now do the differentiation of an arbitrary function F in time by first differentiating at retarded time and then correcting the result via the chain rule.

$$\frac{\partial F_{ret}}{\partial t} = \frac{\partial F_{ret}}{\partial t_{ret}} \frac{\partial t_{ret}}{\partial t} = \left[\frac{1}{1-\vec{\beta}\cdot\hat{r}} \right]_{ret} \dot{F}_{ret} \tag{2.62}$$

We can do the same for the spatial derivatives. We go back to equation (2.60) and replace dt with a spatial derivative:

$$dt_{ret} = -\frac{1}{c}\left[\frac{\vec{r}\cdot\vec{dx}}{r} \right]_{ret} + \left[\vec{\beta}\cdot\hat{r} \right]_{ret} dt_{ret} \tag{2.63}$$

The first term on the right hand side now expresses the change in time of the emission at the source when we shift our position of measurement over a distance dx. If the displacement of dx is *orthogonal* to $\vec{r}$ then this component of dt_{ret} becomes zero: The signal received at x and $x+dx$ was emitted at the same time.

If the displacement along dx is *parallel* to $\vec{r}$, then the time difference becomes maximal and equal to the time needed to move over a distance of dx at the speed of light. We can reorder (2.63) to express the ratio of dt_{ret} and dx:

$$\frac{\partial t_{ret}}{\partial x} = -\frac{1}{c}\left[\frac{\vec{r}_x/r}{(1-\vec{\beta}\cdot\hat{r})} \right]_{ret} \tag{2.64}$$

We can use this relation to differentiate any arbitrary function F with regard to x by first differentiating at retarded time to dt_{ret} and use chain rule with the above result to get the derivative in x. Repeating this for y and z we can write.

$$\nabla\ F_{ret} \ = \ [\ \nabla F\ ]_{ret} \ - \ \frac{1}{c}\left[\ \frac{\hat{r}}{1-\vec{\beta}\cdot\hat{r}}\ \right]_{ret}\ \dot{F}_{ret} \tag{2.65}$$

Using formulas (2.62) and (2.65) we can now derive the electric field E and the magnetic field B from the Liénard Wiechert potentials. Using,

$$\mathsf{E} \ = \ -\nabla\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t} \tag{2.66}$$

$$\Phi = \frac{q}{4\pi\epsilon_0}\left|\frac{1}{r\ (1-\vec{\beta}\cdot\hat{r}\ )}\right|_{ret}, \qquad \vec{A} = \frac{q}{4\pi\epsilon_0}\left|\frac{\vec{\beta}}{r\ (1-\vec{\beta}\cdot\hat{r}\ )}\right|_{ret} \tag{2.67}$$

We get the following expressions for the two terms making up E.

$$\frac{1}{c}\frac{\partial\vec{A}}{\partial t} = \frac{q}{4\pi\epsilon_o}\left(\ \frac{\dot{\beta}/c}{r(1-\vec{\beta}\cdot\hat{r}\ )^2} \ + \ \frac{-\vec{\beta}\cdot\hat{r}+(\dot{\beta}/c)\cdot\vec{r}+\beta^2}{r^2(1-\vec{\beta}\cdot\hat{r}\ )^3}\ \vec{\beta}\ \right)\Bigg|_{ret} \tag{2.68}$$

$$\nabla\Phi \ = \frac{q}{4\pi\epsilon_o}\left(\ \frac{\hat{r}-\vec{\beta}}{r^2(1-\vec{\beta}\cdot\hat{r}\ )^2} \ - \ \frac{-\vec{\beta}\cdot\hat{r}+(\dot{\beta}/c)\cdot\vec{r}+\beta^2}{r^2\ (1-\vec{\beta}\cdot\hat{r}\ )^3}\ \hat{r}\ \right)\Bigg|_{ret} \tag{2.69}$$

Which can be simplified using the substitutions $\lambda = 1/(1-\vec{\beta}\cdot\hat{r}\ )$ for the shockwave factor and $\vec{r}_{ph} = \hat{r}-\vec{\beta}$ for the vector pointing from the phantom position to the measurement point.

$$\frac{1}{c}\frac{\partial\vec{A}}{\partial t} = \frac{q}{4\pi\epsilon_o r^2}\left(\ \lambda^2(r/c)\dot{\beta} \ + \ \lambda^3\left[(r/c)\dot{\beta}\cdot\hat{r} \ - \ \vec{r}_{ph}\cdot\vec{\beta}\ \right]\vec{\beta}\ \right)\Bigg|_{ret} \tag{2.70}$$

$$\nabla\Phi \ = \ \frac{q}{4\pi\epsilon_o r^2}\left(\ \lambda^2\ \vec{r}_{ph} \qquad - \ \lambda^3\left[(r/c)\dot{\beta}\cdot\hat{r} \ - \ \vec{r}_{ph}\cdot\vec{\beta}\ \right]\hat{r}\ \right)\Bigg|_{ret} \tag{2.71}$$

Where $\dot{\beta}$ is related to the acceleration and the velocity of the charge as $\vec{a}/c^2 = \dot{\beta}/c = \dot{v}/c^2$. The time derivative term (2.70) dependents mainly on the acceleration of the charge. This was to be expected since $\vec{A}$ is proportional to the velocity of the charge. The second part of (2.70) depends on the velocity and can be neglected at lower charge velocity. We use a standard vector identity to collect various radiation terms together.

$$A \times (B \times C) \ = \ B \ (A \cdot C) \ - \ C \ (A \cdot B) \tag{2.72}$$

$$\hat{r} \times (\ (\hat{r} - \vec{\beta}) \ \times \dot{\beta} \ ) \ = \ (\hat{r} - \vec{\beta}) \ (\hat{r} \cdot \dot{\beta}) \ - \ \dot{\beta} \ (1 - \vec{\beta} \cdot \hat{r} \ ) \tag{2.73}$$

By doing so we arrive at the usual presentation of electric field from a charge with arbitrary motion and acceleration:

Electric field E of an arbitrary moving/accelarating charge

$$\mathsf{E} \ = \ \frac{q}{4\pi\epsilon_o r^2} \left(\frac{(1-\beta^2) \ (\hat{r} - \vec{\beta})}{(1 - \vec{\beta} \cdot \hat{r} \ )^3} \ + \ \frac{r}{c} \ \frac{\hat{r} \times (\ (\hat{r} - \vec{\beta}) \ \times \dot{\beta} \ )}{(1 - \vec{\beta} \cdot \hat{r} \ )^3} \right) \Bigg|_{ret} \tag{2.74}$$

We can simplify this expression as before by using $\vec{r}_{ph} = \hat{r} - \vec{\beta}$ which is pointing along the direction of the line from the phantom location to the point for which we calculate the field.

$$\mathsf{E} \ = \ \frac{q}{4\pi\epsilon_o r^2} \left(\frac{(1-\beta^2) \ \vec{r}_{ph}}{(\hat{r} \cdot \vec{r}_{ph})^3} \ + \ \frac{r}{c} \ \frac{\hat{r} \times (\ \vec{r}_{ph} \ \times \dot{\beta} \ )}{(\hat{r} \cdot \vec{r}_{ph})^3} \right) \Bigg|_{ret} \tag{2.75}$$

Furthermore we can use the direction dependent shockwave factor $\lambda = 1/(1 - \vec{\beta} \cdot \hat{r} \ )$ which can be interpreted as the "compression" of the emitted potential field in front of the charge due to its motion.

$$\mathsf{E} \ = \ \frac{q}{4\pi\epsilon_o r^2} \ \lambda^3 \ \left(\ (1-\beta^2) \ \vec{r}_{ph} \ + \ \frac{r}{c} \ \hat{r} \times (\ \vec{r}_{ph} \ \times \dot{\beta} \ ) \right) \Bigg|_{ret} \tag{2.76}$$

2.11 Derivation of the Liénard Wiechert B field

For the magnetic B field we need to take the curl $\nabla \times \vec{A}$ of the magnetic vector potential. Careful evaluation shows up that the electric and magnetic field are related to each other by.

$$\mathsf{B} \ = \ \frac{\hat{r}_{ret}}{c^2} \times \mathsf{E} \tag{2.77}$$

Which gives us for the magnetic field of an arbitrary moving and accelerating charge:

Magnetic field B of an arbitrary moving/accelarating charge

$$\mathsf{B} \;=\; \frac{q/c}{4\pi\epsilon_o r^2}\left(\frac{(1-\beta^2)\ \hat{r}\times(\hat{r}-\vec{\beta})}{(1-\vec{\beta}\cdot\hat{r}\ )^3} \;+\; \frac{r}{c}\ \frac{\hat{r}\times(\hat{r}\times(\ (\hat{r}-\vec{\beta})\ \times\dot{\vec{\beta}}\ ))}{(1-\vec{\beta}\cdot\hat{r}\ )^3}\right)\Bigg|_{ret} \tag{2.78}$$

2.12 Point charge radiation fields from acceleration

The point charge radiation fields from acceleration in the z-direction (text to be inserted)

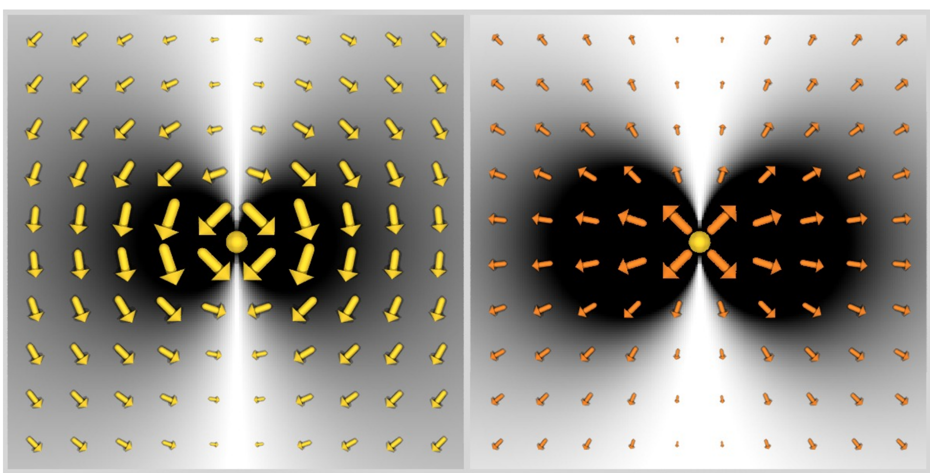

Figure 2.7: Electric and Poynting field of a uniformly accelerating charge

In figure (2.7) It has a direction acceleration opposite to the field which would cause the charge to accelerate. The charge opposes a change in speed. It is this effect which causes self-induction. Multiple charges close together will generate electric radiation fields contrary to the accelerating electric field. The ensemble is harder to accelerate when the charges are closer together and easier when they are further apart.

$$\mathsf{E}_{acc} = \frac{q}{4\pi\epsilon_0}\frac{a_z}{c^2}\left\{ \frac{xz}{r^3}, \quad \frac{yz}{r^3}, \quad -\frac{x^2+y^2}{r^3} \right\} \tag{2.79}$$

$$\mathsf{B}_{acc} = \frac{q}{4\pi\epsilon_0}\frac{a_z}{c^3}\left\{ -\frac{y}{r^2}, \quad \frac{x}{r^2}, \quad 0 \right\} \tag{2.80}$$

$$\vec{\mathcal{P}}_{acc} = \frac{q^2}{16\pi^2\epsilon_0}\frac{a_z^2}{c^3}\left\{ \frac{x^2+y^2}{r^5}x, \quad \frac{x^2+y^2}{r^5}y, \quad \frac{x^2+y^2}{r^5}z \right\} \tag{2.81}$$